GROUP THEORY 2024 - 25, SOLUTION SHEET 5

Exercise 1. To do yourself. Ask the assistant if something is unclear.

Exercise 2. (1) Any element in a finite group is torsion, hence $Tor(A) = A$.

(2) No element except 0 have finite order, so its torsion group is trivial.

(3) Let $[q] \in \mathbb{Q}/\mathbb{Z}$ be any element represented by $q = \frac{a}{b} \in \mathbb{Q}$. Then
$$b[q] = [bq] = [a] = [0] \in \mathbb{Q}/\mathbb{Z}$$
since $a \in \mathbb{Z}$. Hence every element is torsion and $Tor(\mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$.

(4) Let $x \in \mathbb{C}^\times$ and write it in polar form $x = re^{i\theta}$ with $r > 0$ and $\theta \in [0, 2\pi)$. Then $x^n = r^n e^{in\theta} = 1$ if and only if $r = 1$ and $n\theta = 0 \mod 2\pi$, i.e. $x = e^{2\pi ik/n}$ for $k \in \mathbb{Z}$. Those are the n-roots of unity μ_n. Hence
$$Tor(\mathbb{Q}^\times) = \mu_\infty = \bigcup_{n\in\mathbb{N}_{>0}} \mu_n.$$

(5) We know that subgroups of $\mathbb{Z}$ are of the form $n\mathbb{Z} \cong \mathbb{Z}$ which are free, hence without torsion.

(6) We saw in the course that subgroups of finite free abelian group are free abelian, which shows that their torsion subgroup is trivial.

Exercise 3. Since G is finitely generated, there exists a finite set of generators for G. Let $g_1, g_2, \ldots, g_k$ be a set of generators for G, so that every element of G can be written as an integer linear combination of these generators:
$$g = n_1g_1 + n_2g_2 + \cdots + n_kg_k,$$
where $a_1, a_2, \ldots, a_k \in \mathbb{Z}$.

Since $\mathrm{Tor}(G) = G$, every element in G is a torsion element. This implies that for each generator $g_i \in G$, there exists a positive integer m_i minimal with the property that $m_i \cdot g_i = 0$ for $i = 1, \ldots, k$ (m_i is the order of g_i).

Since G is generated by the finite set $\{g_1, g_2, \ldots, g_k\}$ and each g_i has finite order m_i, there are only finitely many possible combinations of the generators $g_1, g_2, \ldots, g_k$ with integer coefficients a_i modulo m_i, implying that G itself is finite.

Exercise 4. **(1) $\implies$ (2)**: For all $i \in I$, define $e_i \in \mathbb{Z}^{\oplus I}$ as:

$$e_i := (a_j)_{j\in I} \in \mathbb{Z}^{\oplus I}, \text{ where } a_j = 1 \text{ if } j = i \text{ and } a_j = 0 \text{ if } j \neq i.$$

It is straightforward to show using the definition of direct sums that the set $\{e_i\}_{i\in I}$ is a basis for $\mathbb{Z}^{\oplus I}$. Now if $A \cong \mathbb{Z}^{\oplus I}$ the then homomorphic image of the e_i for a basis for A.

(2) $\implies$ (1): Fix a basis, $(a_k)_{k\in I}$ of A, then every element $x \in A$ can be uniquely written as:

$$x = \sum_{k\in I} n_k a_k$$

for some $n_k \in \mathbb{Z}$. Consider the following function, which is well-defined due to the aforementioned uniqueness:

$$\varphi : A \to \mathbb{Z}^{\oplus I}, \sum_{k\in I} n_k a_k \mapsto (n_k)_{k\in I}.$$

It is a straightforward check to see that φ is an isomorphism of Abelian groups.

Exercise 5. (1) Suppose first that G is free abelian. The previous exercise tells us that there exists a set I and a basis $B = \{a_i | i \in I\} \subset G$ such that all elements $x \in G$ can be uniquely written as finite sums

$$x = \sum_{k\in I} n_k a_k$$

where all but finitely many n_k equal 0. Let A be another abelian group with a set function $f : B \to A$. Let us prove the existence of φ by defining

$$\varphi : G \to A$$
$$x = \sum_{k\in I} n_k a_k \mapsto \sum_{k\in I} n_k f(a_k)$$

This is well defined since all but finitely many n_k are non-zero (it is a finite sum). It is clearly a group homomorphism (to check for yourself) and $\varphi(i(a_k)) = \varphi(a_k) = f(a_k)$ for all $a_k \in B$ so that $\varphi \circ i = f$. To prove unicity, suppose that there exists two homomorphism $\varphi, \varphi' : G \to A$ extending f. Then for all $x = \sum_{k\in I} n_k a_k \in G$ we have

$$\begin{aligned}
\varphi'(x) = \varphi'(\sum_{k\in I} n_k a_k) &= \sum_{k\in I} n_k \varphi'(a_k) \\
&= \sum_{k\in I} n_k f(a_k) \\
&= \sum_{k\in I} n_k \varphi(a_k) = \varphi(\sum_{k\in I} n_k a_k) = \varphi(x)
\end{aligned}$$

which proves that $\varphi = \varphi'$ (we used the fact that both φ and φ' are linear and extend f).

(2) Suppose now that G satisfies the universal property of free abelian groups. We will show that G is indeed free abelian by showing that $G \cong \mathbb{Z}^{\oplus B}$ for B the set given by the universal property of G. Note that the idea of the following proof is always used when dealing with universal properties, which you will (probably) encounter again in the future.

Let $f : B \to \mathbb{Z}^{\oplus B}$ given by the inclusion of the basis, i.e. $f : b \mapsto e_b \in \mathbb{Z}^{\oplus B}$ defined in the proof of the previous exercise (e_b is a generalization of $e_i \in k^n$ of linear algebra). Using the universal property of G, the map f extend to a morphism $\varphi : G \to \mathbb{Z}^{\oplus B}$ such that the following triangle commutes

$$\begin{array}{ccc} B & \xrightarrow{\ i\ } & G \\ & \searrow^{f} & \downarrow{\scriptstyle\varphi} \\ & & \mathbb{Z}^{\oplus B}. \end{array}$$

Since $\mathbb{Z}^{\oplus B}$ is free abelian with basis $f : B \subset \mathbb{Z}^{\oplus B}$, it satisfies the universal property (proved in the first point) of free abelian groups. Hence we can extend $i : B \to G$ along $f : B \to \mathbb{Z}^{\oplus B}$ to obtain $\varphi' : \mathbb{Z}^{\oplus B} \to G$ such that the following triangle commutes:

$$\begin{array}{ccc} B & \xrightarrow{\ f\ } & \mathbb{Z}^{\oplus B} \\ & \searrow^{i} & \downarrow{\scriptstyle\varphi'} \\ & & G. \end{array}$$

We now prove that φ and φ' are inverse of each other by applying two more times the universal property of free abelian groups. First we apply it to G and it tells us that there exists a unique $\psi : G \to G$ such that the following triangle commutes:

$$\begin{array}{ccc} B & \xrightarrow{\ i\ } & G \\ & \searrow^{i} & \downarrow{\scriptstyle\psi} \\ & & G. \end{array}$$

Since the identity $Id_G : G \to G$ does the job, any such ψ must be the identity. But for $\psi = \varphi' \circ \varphi$ we have that $\psi \circ i = \varphi' \circ \varphi \circ i = \varphi' \circ f = i$, where we use the commutativity of the two first triangles. As explained, by unicity of such maps, we must have that $\varphi' \circ \varphi = Id_G$. In a similar fashion, we can use the universal property of $\mathbb{Z}^{\oplus B}$ to show that $\varphi \circ \varphi' = Id_{\mathbb{Z}^{\oplus B}}$. This shows that $G \cong \mathbb{Z}^{\oplus B}$ which ends the proof.

Exercise 6. Since F is free with basis $\{e_1, e_2, e_3\}$ we can apply the universal property of exercise 5 with $B = \{e_1, e_2, e_3\}$, $G = F$ and $A = \mathbb{Z}^2$. It tells us that there exists a unique group homomorphism $\varphi : F \to \mathbb{Z}^2$ which extends f. The image of a group homomorphism is always a subgroup of the codomain. Since we saw in the lectures that subgroups of finite free abelian groups are finite free abelian, this answers positively to the question.

Exercise 7. We will constantly use the fact that any subgroup of $\mathbb{Z}^k$ is free of rank $l \leq k$. In each case we will denote the Abelian group in question by A.

(1) Since $\{(1,1)\}$ is a generating set of A and is linearly independent, it is a basis for A and hence the rank of A is 1.
(2) The rank of A is 1 again since $B = \{(1,2)\}$ is a basis for A. The set B is linearly independent and generates A as $(-3,-6) = (-3)(1,2)$.
(3) One checks that $\{1, \sqrt{2}, \sqrt{3}\}$ forms a basis for A and hence the rank of A is 3.
(4) The rank of A is 3 since the three elements generate A and are linearly independent which can be seen by observing that the determinant of the following matrix is non-zero:

$$\begin{pmatrix} 1 & 2 & 1 \\ 5 & 3 & -9 \\ 1 & 8 & 34 \end{pmatrix}.$$

(5) Note that the set $B = \{(1,5,1),(2,3,8)\}$ is linearly independent and generates A since $(1,-9,13) = (-3)(1,5,1) + 2(2,3,8)$. Hence the rank of A is 2.

Exercise 8. Let $\varphi : \mathbb{Z}^n \to \mathbb{Z}^m$ be an isomorphism of abelian groups. Fix a prime number p and consider the following subgroup of $\mathbb{Z}^m$:

$$H := \{(a_1, ..., a_m) \in \mathbb{Z}^m \mid a_i \in p\mathbb{Z}\}.$$

Note that $\mathbb{Z}^m/H \cong (\mathbb{Z}/p\mathbb{Z})^m$.

We leave it as a little exercise to the reader to show that

$$\varphi^{-1}(H) = \{(a_1, ..., a_n) \in \mathbb{Z}^n \mid a_i \in p\mathbb{Z}\}$$

and $\mathbb{Z}^n/\varphi^{-1}(H) \cong (\mathbb{Z}/p\mathbb{Z})^n$.

Since φ is in particular a surjective homomorphism, the correspondence theorem along with the third isomorphism theorem implies that φ induces an isomorphism:

$$\overline{\varphi} : \mathbb{Z}^n/\varphi^{-1}(H) \to \mathbb{Z}^m/H.$$

Hence we have an isomorphism of abelian groups:

$$\overline{\varphi} : (\mathbb{Z}/p\mathbb{Z})^n \to (\mathbb{Z}/p\mathbb{Z})^m.$$

Which automatically is $\mathbb{Z}/p\mathbb{Z}$ - linear since it a morphism of abelian groups. Since isomorphic vector spaces must have the same dimension we obtain that $m = n$.

Exercise 9. The same proof as Proposition 11 of the lecture notes apply to show that $\mathbb{Q}^{>0}$ is not finitely generated. To show that it is free, we show that the set $B = \{p_i |\ p_i \text{ is prime}\}$ of prime numbers forms a basis. Let $q = \frac{a}{b}$ be written in indecomposable form, with $a, b \in \mathbb{N}_*$. Decompose a and b as a product of powers of prime numbers. Note that the prime numbers appearing in each decomposition are distinct since the fraction $\frac{a}{b}$ has been chosen to be indecomposable. Using those decompositions, we obtain q as a finite product of powers of elements of B (the powers are negative for the primes appearing in the decomposition of b). If there was more than one decomposition of q as a product of powers of primes, it would yield distinct decompositions of either a or b (or of both) as product of powers of primes, by seperating the positive and negative powers. By unicity of the decomposition of natural numbers (seen in linear algebra 2) we obtin a contradiction.

We have shown that B is a basis of the abelian group $\mathbb{Q}^{>0}$, which means that it is free by exercise 4.

Exercise 10. We refer to the diagram in the exercise sheet for notation. Suppose F is a finitely generated free Abelian group, then fix a basis $e_1, e_2, ..., e_n$ for F. Since ϕ is surjective we can choose pre-images $g_1, ..., g_n$ in G of $\psi(e_1), ..., \psi(e_n)(H)$. It follows from the universal property of free Abelian groups that we can define a map $\alpha : A \to G$, making the diagram commute by simply sending e_i to g_i. Hence A is projective.

Conversely suppose that A is a finitely generated Abelian group, then let $a_1, ..., a_n$ be any generating set. Then we obtain a surjective group homomorphism $\phi : \mathbb{Z}^n \to A$ which is defined by sending the usual basis e_i to a_i. Let K be the kernel of the homomorphism ϕ. Now K is a free Abelian group since we know from the lectures that subgroups of finitely generated free Abelian groups are free. Hence we obtain a short exact sequence:

$$0 \to K \to \mathbb{Z}^n \xrightarrow{\phi} A \to 0.$$

Let $\psi : A \to A$ be the identity map, the the projectivity of A implies that there exists a map $\alpha : A \to \mathbb{Z}^n$ such that $\phi \circ \alpha = Id_A$. Hence the above exact sequence splits on the right. Therefore A is a subgroup of $\mathbb{Z}^n$ and is hence a free Abelian group.

Exercise 11. Consider the short exact sequence:

$$0 \to 2\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

The induced sequence of torsion subgroups is:

$$0 \to 0 \to 0 \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Which is clearly not exact due to the failure of the surjectivity of the map $0 \to \mathbb{Z}/2\mathbb{Z}$.